Definition: $f_n : E \mapsto E'$, $f : E \mapsto E'$.

(1) f_n converges at $p_0 \in E$ if $f_n(p_0)$ converges on E' $\forall n \in \mathbb{N}$.

(2) f_n converges on E if at each $p \in E$, f_n converges pointwisely on E'.

(3) $f(p) = \lim_{n\to\infty} f_n(p)$, f_n converges to f.

and write $f = \lim_{n\to\infty} f_n$.

Example (1) $\lim_{n\to\infty} x - \frac{x}{n} = x$

$$\left.\begin{aligned} f_n(x) &= x - \frac{x}{n} \\ f(x) &= x \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} f_n(x) = f(x),$$

(2) $f_n(x) = x^n$, $f_n : [0,1] \mapsto \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

<u>Definition</u>: Let $f_n: E \mapsto E'$, $f: E \mapsto E'$

f_n converges to f uniformly if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $d(f_n(x), f(x)) < \varepsilon$ $\forall x \in E$.

<u>Example</u> $f_n(x) = x - \frac{x}{n}$, $f_n: [0,1] \mapsto \mathbb{R}$

$f(x) = x$, $f: [0,1] \mapsto \mathbb{R}$

$$d(f_n(x), f(x)) = |f_n(x) - f(x)| = |x - \tfrac{x}{n} - x|$$

$$= \left|\frac{x}{n}\right| = \frac{x}{n} \le \frac{1}{n} < \varepsilon$$

if $n \geqslant \lfloor \frac{1}{\varepsilon} \rfloor + 1 = N$

f_n converges to f uniformly.

<u>Observation</u> Uniform convergence $\Rightarrow$ pointwise convergence.

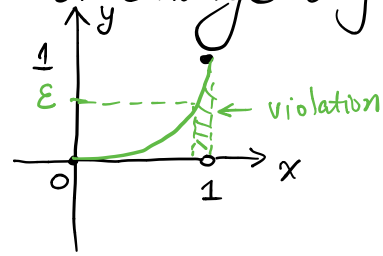
<u>Example</u> Show $f_n(x) = x - \frac{x}{n}$, $f_n: \mathbb{R} \mapsto \mathbb{R}$ does not converges uniformly.

~~proof~~ Let $\varepsilon > 0$, $N \in \mathbb{N}$.

$$|f_n(x) - f(x)| = \frac{|x|}{N} > \varepsilon \quad \text{if } |x| > \varepsilon N.$$

<u>Example</u> $f_n(x) = x^n$, $f_n : [0,1] \mapsto \mathbb{R}$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$ does not converge uniformly.



~~proof~~: $d(f_n(\tfrac{1}{2}), f(\tfrac{1}{2})) =$

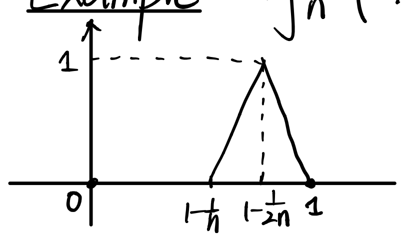
$$\left| f_n\left(\left(\tfrac{1}{2}\right)^{\frac{1}{N}}\right) - f\left(\left(\tfrac{1}{2}\right)^{\frac{1}{N}}\right) \right|$$

$$= \frac{1}{2} > \varepsilon \quad \text{if} \quad \varepsilon < \frac{1}{2} \ \forall n > N$$

$\Rightarrow$ $f_n \not\to f$ uniformly.

<u>Example</u>

$$f_n(x) = \begin{cases} 0, & 0 \le x \le 1 - \frac{1}{n} \\ \dfrac{x - (1 - \frac{1}{n})}{1/2n}, & \text{if } 1 - \frac{1}{n} < x < 1 - \frac{1}{2n} \\ \dfrac{1 - x}{1/2n}, & \text{if } 1 - \frac{1}{2n} \le x \le 1 \end{cases}$$



$f_n(x) \to x = 0$ pointwisely but not uniformly.

~~Proposition~~ $f_n : E \mapsto E'$, E' is complete, then f_n converges uniformly $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $m, n > N$ implies $d(f_n(x), f_m(x)) < \varepsilon$ $\forall x \in E$.

~~proof~~ $\Rightarrow$) Assume $f_n \to f$ uniformly.

Let $\varepsilon > 0$, $N > 0$ s.t. $k > N$ $d(f_k(x), f(x)) < \frac{\varepsilon}{2}$

$$\Rightarrow d(f_n(x), f_m(x)) \leq d(f_n(x), f(x)) + d(f(x), f_m(x))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Leftarrow$) $\forall \varepsilon > 0$, $\exists N$ s.t. $m, n > N$ implies $d(f_m(x_0), f_m(x_0)) < \varepsilon$, for a fixed $x_0 \in E$.

$\Rightarrow$ $f_n(x_0)$ is a Cauchy sequence in E',

Since E' is complete. $f_n(x_0)$ converges to a unique point in E' denote as $f(x_0)$.

For any point $x \in E$, we can find a point $f(x)$, s.t. $\lim_{n \to \infty} f_n(x) = f(x)$. $f_n \to f$ pointwisely.

Let $\varepsilon > 0$, $\exists N$ s.t. $m, n > N \Rightarrow$

$d(f_m(x), f_n(x)) < \varepsilon/2 \ \forall x \in E$.

Since $f_m(x) \to f(x)$, $\exists N_x$ s.t. $m > N_x \Rightarrow$

$d(f_m(x), f(x)) < \varepsilon/2 \quad \forall x \in E$.

$$\Rightarrow \quad d(f_n(x), f(x)) \le d(f_n(x), f_m(x)) + d(f_m(x), f(x))$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $f_n \to f$ uniformly.

<u>Theorem</u> $f_n : E \mapsto E'$, f_n continuous.

$f_n \to f$ uniformly $\Rightarrow f$ is continuous.

<u>proof</u>: Let $x_0 \in E$, $\varepsilon > 0$

$$d(f(x), f(x_0)) \le d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)). \quad (*)$$

Select N such that $d(f_N(y), f(y)) < \frac{\varepsilon}{3}$ $\forall y \in E$ since $f_n \to f$.

Since f_N is continuous, $\exists\, \delta > 0$,

$$d(x, x_0) < \delta \Rightarrow d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}.$$

Then, if $d(x, x_0) < \delta$

$$d(f(x), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

from equation $(*)$.